Math 304 Sample Final

Name: ____________________

This exam has 11 questions, for a total of 150 points.

Please answer each question in the space provided. Please write **full solutions**, not just answers. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	20	
7	20	
8	15	
9	10	
10	15	
11	15	
Total:	150	

Question 1. (10 pts)

Determine the following statements are true or false.

(a) If B is diagonalizable, then B^2 is also diagonalizable.

(b) If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ span $\mathbb{R}^8$, then n must be 8.

(c) If $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linear independent in $\mathbb{R}^7$, then n must be at most 7.

(d) Suppose that A is a diagonalizable 5×5 matrix. If $A^5 = 0$, then A must be the zero matrix.

(e) Suppose $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ form a basis of $\mathbb{R}^3$, then $\vec{v}_1$, $(\vec{v}_1 + \vec{v}_2)$ and $(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$ also form a basis of $\mathbb{R}^3$.

Question 2. (15 pts)

(a) Let A be a (4×4) matrix. We view A as a linear mapping $\mathbb{R}^4 \to \mathbb{R}^4$. Suppose $\det(A) \neq 0$. What is the dimension of the range of A? Justify your answer.

(b) Suppose $F : \mathbb{R}^5 \to \mathbb{R}^3$ is linear mapping. Is it possible that $\dim \ker F = 0$? Justify your answer.

(c) Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping. What are the possible integer values that $\dim(\ker B)$ can take? List all possibilities.

Question 3. (10 pts)

Find the determinant of

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Question 4. (10 pts)

Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 2-i \\ 2+i & 0 \end{bmatrix}$$

Question 5. (10 pts)

Given the vectors

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 5 \\ 8 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^4$. Determined whether $v = (3, 0, 1, 4)^T$ is in the span of u_1, u_2, u_3.

Question 6. (20 pts)

Given the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

(a) Find the reduced row echelon form of A.

(b) Find a basis for the kernel of A.

(c) Find a basis for the range of A.

Question 7. (20 pts)

Determine whether the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable. If yes, find a nonsingular matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

Question 8. (15 pts)

Let V be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Use Gram-Schmidt process to find an orthonormal basis of V.

(b) Find the projection of

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

onto V.

Question 9. (10 pts)

Let $\mathbb{P}_2 = \{\text{all polynomials of degree } \leq 2\}$. We define the following inner product on $\mathbb{P}_2$:

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Find the angle between $x + 1$ and x^2.

Question 10. (15 pts)

Let $V = \mathrm{span}(e^x, e^{-x}, xe^x, xe^{-x})$ be a subspace of C^∞. We know that V is a vector space with a basis

$$e^x,\ e^{-x},\ xe^x,\ xe^{-x}.$$

Let us denote this basis by $\mathfrak{B}$. Let

$$T(f) = 2f - f'$$

be a linear transformation from V to V .

(a) Find the matrix representation of T with the basis $\mathfrak{B}$.

(b) Is T nonsingular? Justify your answer.

Question 11. (15 pts)

The eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ are $\lambda_1 = 1$ with $v_1 = (1, 0, 0)^T$, $\lambda_2 = 2$ with $v_2 = (1, 1, 0)^T$ and $\lambda_3 = 3$ with $v_3 = (1, 2, 1)^T$.

(a) Find the general solution to the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

(b) Find a specific solution $\mathbf{x}(t)$ such that

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

when $t = 0$.